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Technical Comments

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Comment on "A Unified Mathematical Framework for Strapdown Algorithm Design"

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In the above-mentioned paper [1], Savage described a unified mathematical framework for strapdown inertial navigation algorithm design. Quite recently, the present author also proposed a dual-quaternion representation of the principle of strapdown inertial navigation [2]. This Comment makes significant simplifications to the differential equation of the "velocity translation vector" (VTV) and shows the underlying connections between them.

The "translation" in the context of the dual-quaternion approach is a local concept between two considered frames (see footnote 4 in [2]). It could be velocity or position depending on the specific frames under investigation, an interesting coincidence with Savage's description. Founded on the screw theory, Eq. (65) in [2] is a general result in the sense of being valid for any local concept, whether it is velocity or position.

Noticing the identity $\boldsymbol{\sigma}\times(\boldsymbol{\sigma}\times\boldsymbol{\sigma}')=\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}')-\sigma^2\boldsymbol{\sigma}'$, it can be rewritten as

$$\begin{aligned}\Delta\mathbf{t}^{N'} &= \boldsymbol{\sigma}'\frac{\sin(\sigma)}{\sigma}+\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}')\cdot\boldsymbol{\sigma}}{\sigma^2}\left(1-\frac{\sin(\sigma)}{\sigma}\right)\\ &\quad+2\boldsymbol{\sigma}\times\boldsymbol{\sigma}'\left(\frac{\sin(\sigma/2)}{\sigma}\right)^2=\boldsymbol{\sigma}'+\frac{\boldsymbol{\sigma}\times(\boldsymbol{\sigma}\times\boldsymbol{\sigma}')}{\sigma^2}\left(1-\frac{\sin(\sigma)}{\sigma}\right)\\ &\quad+\boldsymbol{\sigma}\times\boldsymbol{\sigma}'\left(\frac{1-\cos(\sigma)}{\sigma^2}\right)=\left[I+\frac{1-\cos(\sigma)}{\sigma^2}(\boldsymbol{\sigma}\times)\right.\\ &\quad\left.+\frac{1}{\sigma^2}\left(1-\frac{\sin(\sigma)}{\sigma}\right)(\boldsymbol{\sigma}\times)^2\right]\boldsymbol{\sigma}' \end{aligned}\tag{1}$$

which, regardless of the notational difference, is exactly the second equation of Eq. (8) in [1]. So the implicitly defined VTV is identical to the dual part of an appropriate screw vector in the dual-quaternion representation. Note that the screw vector is a well-established term in the literature, for example, [3,4]. The Euler vector coming from the Euler theorem, the screw vector derives itself from the Chasles theorem [5] stating that "the general displacement of a rigid body in space consists of a rotation about an axis (the screw axis) and a translation parallel to that axis". By the Principle of Transference [6], the differential equation of the screw vector takes exactly the same form as that of the Euler vector.

The differential equation for VTV can be derived from the differential equation of an appropriate screw vector. Considering the inertial frame and the thrust velocity frame in [2], we have

$$\hat{\boldsymbol{\sigma}}=\boldsymbol{\sigma}+\varepsilon\boldsymbol{\sigma}',\qquad \hat{\sigma}=\sigma+\varepsilon\sigma' \tag{2}$$

$$\hat{\boldsymbol{\omega}}=\boldsymbol{\omega}+\varepsilon\mathbf{s} \tag{3}$$

$$\tfrac{1}{2}\hat{\boldsymbol{\sigma}}\times\hat{\boldsymbol{\omega}}=\tfrac{1}{2}(\boldsymbol{\sigma}\times\boldsymbol{\omega})+\varepsilon\tfrac{1}{2}(\boldsymbol{\sigma}\times\mathbf{s}+\boldsymbol{\sigma}'\times\boldsymbol{\omega}) \tag{4}$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}+\varepsilon 2\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'=\hat{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{\sigma}}=\hat{\sigma}^2=\sigma^2+\varepsilon 2\sigma\sigma' \tag{5}$$

$$\begin{aligned}\frac{1}{\hat\sigma^2}\left(1-\frac{\hat\sigma\sin\hat\sigma}{2(1-\cos\hat\sigma)}\right)&=\left(\frac{1}{\sigma^2}-\varepsilon\frac{2\sigma'}{\sigma^3}\right)\left[1-\frac{\sigma\sin\sigma}{2(1-\cos\sigma)}\right.\\ &\quad\left.-\varepsilon\frac{\sigma'\sin\sigma-\sigma\sigma'}{2(1-\cos\sigma)}\right]=\frac{1}{\sigma^2}\left(1-\frac{\sigma\sin\sigma}{2(1-\cos\sigma)}\right)\\ &\quad-\varepsilon\left[\frac{\sigma'\sin\sigma-\sigma\sigma'}{2\sigma^2(1-\cos\sigma)}+\frac{2\sigma'}{\sigma^3}\left(1-\frac{\sigma\sin\sigma}{2(1-\cos\sigma)}\right)\right]\\ &=\frac{1}{\sigma^2}\left(1-\frac{\sigma\sin\sigma}{2(1-\cos\sigma)}\right)+\varepsilon\left[\frac{\sigma'\sin\sigma+\sigma\sigma'}{2\sigma^2(1-\cos\sigma)}-\frac{2\sigma'}{\sigma^3}\right]\\ &=\frac{1}{\sigma^2}\left(1-\frac{\sigma\sin\sigma}{2(1-\cos\sigma)}\right)+\varepsilon\left[\frac{\sin\sigma+\sigma}{2\sigma^3(1-\cos\sigma)}-\frac{2}{\sigma^4}\right]\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\end{aligned}\tag{6}$$

$$\begin{aligned}\hat{\boldsymbol{\sigma}}\times(\hat{\boldsymbol{\sigma}}\times\hat{\boldsymbol{\omega}})&=\boldsymbol{\sigma}\times(\boldsymbol{\sigma}\times\boldsymbol{\omega})+\varepsilon[\boldsymbol{\sigma}\times(\boldsymbol{\sigma}\times\mathbf{s})+\boldsymbol{\sigma}\\ &\quad\times(\boldsymbol{\sigma}'\times\boldsymbol{\omega})+\boldsymbol{\sigma}'\times(\boldsymbol{\sigma}\times\boldsymbol{\omega})]\end{aligned}\tag{7}$$

The differential equation of VTV is obtained by extracting the dual part from Eq. (35) in [2]

$$\dot{\hat{\boldsymbol{\sigma}}}=\hat{\boldsymbol{\omega}}+\frac{1}{2}\hat{\boldsymbol{\sigma}}\times\hat{\boldsymbol{\omega}}+\frac{1}{\hat\sigma^2}\left(1-\frac{\hat\sigma\sin\hat\sigma}{2(1-\cos\hat\sigma)}\right)\hat{\boldsymbol{\sigma}}\times(\hat{\boldsymbol{\sigma}}\times\hat{\boldsymbol{\omega}}) \tag{8}$$

that is,

$$\begin{aligned}\dot{\boldsymbol{\sigma}}'&=\mathbf{s}+\frac{1}{2}(\boldsymbol{\sigma}\times\mathbf{s}+\boldsymbol{\sigma}'\times\boldsymbol{\omega})+\frac{1}{\sigma^2}\left(1-\frac{\sigma\sin\sigma}{2(1-\cos\sigma)}\right)\\ &\quad\times[\boldsymbol{\sigma}\times(\boldsymbol{\sigma}\times\mathbf{s})+\boldsymbol{\sigma}\times(\boldsymbol{\sigma}'\times\boldsymbol{\omega})+\boldsymbol{\sigma}'\times(\boldsymbol{\sigma}\times\boldsymbol{\omega})]\\ &\quad+\left[\frac{\sin\sigma+\sigma}{2\sigma^3(1-\cos\sigma)}-\frac{2}{\sigma^4}\right](\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}')\boldsymbol{\sigma}\times(\boldsymbol{\sigma}\times\boldsymbol{\omega})\end{aligned}\tag{9}$$

which (the coefficient $\{[(\sin\sigma+\sigma)/2\sigma^3(1-\cos\sigma)]-2/\sigma^4\}$ approaches $1/360$ as $\sigma\to 0$) is much simpler than $\dot{\boldsymbol{\eta}}$ in Eq. (15) in [1]. In fact, the formulation of $\dot{\boldsymbol{\eta}}$ there can be analytically reduced to Eq. (9). See the Appendix for details. Admittedly, without the target form of Eq. (9) in sight, the reduction of $\dot{\boldsymbol{\eta}}$ would have been difficult and possibly unsuccessful.

We have revisited the VTV and simplified its rate equation with an eye for the dual-quaternion approach. The derivations are straightforward from screw theory. In fact, as far as rotation/velocity

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updating is concerned, one could just deal with the differential equations of screw vectors [Eq. (35) in [2]], which are as simple as the differential equation of the Euler vector. As shown in [2], we can design one single algorithm to solve all of the dual-quaternion rate equations involved, using the traditional two-speed approach originally developed in attitude integration. An overwhelming advantage is that the algorithm integrates both conventional coning/sculling corrections in Savage's framework into a so-called "screw correction" and one does not have to design updating algorithms for attitude/velocity, respectively, (See Sec. VI in [2]).

Appendix: Proof of Equivalence of Eq. (9) and $\dot{\eta}$ in Eq. (15) in [1]

With the differential equation of the rotation vector in Eq. (15) in [1] and the identity $(\boldsymbol{\phi}\times)^3 = -\phi^2(\boldsymbol{\phi}\times)$,

$$\dot{\boldsymbol{\phi}} = \boldsymbol{\omega} + \tfrac{1}{2}\boldsymbol{\phi}\times\boldsymbol{\omega} + f_5\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\omega})$$

$$\dot{\boldsymbol{\phi}}\times\boldsymbol{\eta} = (1 - f_5\phi^2)\boldsymbol{\omega}\times\boldsymbol{\eta} + \tfrac{1}{2}(\boldsymbol{\phi}\times\boldsymbol{\omega})\times\boldsymbol{\eta} + f_5(\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\times\boldsymbol{\eta}$$

$$\boldsymbol{\phi}\times(\dot{\boldsymbol{\phi}}\times\boldsymbol{\eta}) = (1 - f_5\phi^2)\boldsymbol{\phi}\times(\boldsymbol{\omega}\times\boldsymbol{\eta}) + \tfrac{1}{2}(\boldsymbol{\phi}\cdot\boldsymbol{\eta})\boldsymbol{\phi}\times\boldsymbol{\omega} + f_5(\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\eta})$$

$$\boldsymbol{\phi}\times[\boldsymbol{\phi}\times(\dot{\boldsymbol{\phi}}\times\boldsymbol{\eta})] = (1 - f_5\phi^2)[(\boldsymbol{\phi}\cdot\boldsymbol{\eta})\boldsymbol{\phi}\times\boldsymbol{\omega} - (\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\times\boldsymbol{\eta}] + \tfrac{1}{2}(\boldsymbol{\phi}\cdot\boldsymbol{\eta})\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\omega}) - \phi^2 f_5(\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\times\boldsymbol{\eta} \tag{A1}$$

$$\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}} = \boldsymbol{\phi}\times\boldsymbol{\omega} + \tfrac{1}{2}\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\omega}) - \phi^2 f_5\boldsymbol{\phi}\times\boldsymbol{\omega}$$

$$(\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}})\times\boldsymbol{\eta} = (1 - f_5\phi^2)(\boldsymbol{\phi}\times\boldsymbol{\omega})\times\boldsymbol{\eta} + \tfrac{1}{2}[(\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\times\boldsymbol{\eta} - \phi^2\boldsymbol{\omega}\times\boldsymbol{\eta}]$$

$$\boldsymbol{\phi}\times[(\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}})\times\boldsymbol{\eta}] = (1 - f_5\phi^2)(\boldsymbol{\phi}\cdot\boldsymbol{\eta})\boldsymbol{\phi}\times\boldsymbol{\omega} + \tfrac{1}{2}[(\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\eta}) - \phi^2\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\eta})],$$

$$\boldsymbol{\phi}\times\{\boldsymbol{\phi}\times[(\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}})\times\boldsymbol{\eta}]\} = (1 - f_5\phi^2)(\boldsymbol{\phi}\cdot\boldsymbol{\eta})\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\omega}) + \tfrac{1}{2}\{-(\boldsymbol{\phi}\cdot\boldsymbol{\omega})\phi^2\boldsymbol{\phi}\times\boldsymbol{\eta} - \phi^2[(\boldsymbol{\phi}\cdot\boldsymbol{\eta})\boldsymbol{\phi}\times\boldsymbol{\omega} - (\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\times\boldsymbol{\eta}]\}$$

Substituting into Eq. (15) in [1], the differential equation for VTV is

$$\begin{aligned}\dot{\boldsymbol{\eta}} &= \mathbf{a}_{SF} + \tfrac{1}{2}\boldsymbol{\phi}\times\mathbf{a}_{SF} + f_5\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\mathbf{a}_{SF}) - (\tfrac{1}{2} - f_4\phi^2)(1 - f_5\phi^2)\boldsymbol{\omega}\times\boldsymbol{\eta} - \tfrac{1}{2}(\tfrac{1}{2} - f_4\phi^2)(\boldsymbol{\phi}\times\boldsymbol{\omega})\times\boldsymbol{\eta} - (\tfrac{1}{2} - f_4\phi^2)f_5(\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\times\boldsymbol{\eta} \\ &\quad - (f_5 - f_8\phi^2)(1 - f_5\phi^2)\boldsymbol{\phi}\times(\boldsymbol{\omega}\times\boldsymbol{\eta}) - \tfrac{1}{2}(f_5 - f_8\phi^2)(\boldsymbol{\phi}\cdot\boldsymbol{\eta})\boldsymbol{\phi}\times\boldsymbol{\omega} - (f_5 - f_8\phi^2)f_5(\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\eta}) \\ &\quad + f_3(1 - f_5\phi^2)(\boldsymbol{\phi}\times\boldsymbol{\omega})\times\boldsymbol{\eta} + \tfrac{1}{2}f_3[(\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\times\boldsymbol{\eta} - \phi^2\boldsymbol{\omega}\times\boldsymbol{\eta}] - \tfrac{1}{2}f_3(1 - f_5\phi^2)(\boldsymbol{\phi}\cdot\boldsymbol{\eta})\boldsymbol{\phi}\times\boldsymbol{\omega} - \tfrac{1}{4}f_3[(\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\eta}) \\ &\quad - \phi^2\boldsymbol{\phi}\times(\boldsymbol{\omega}\times\boldsymbol{\eta})] + f_6(1 - f_5\phi^2)[(\boldsymbol{\phi}\cdot\boldsymbol{\eta})\boldsymbol{\phi}\times\boldsymbol{\omega} - (\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\times\boldsymbol{\eta}] + \tfrac{1}{2}f_6(\boldsymbol{\phi}\cdot\boldsymbol{\eta})\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\omega}) - \phi^2 f_5 f_6(\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\times\boldsymbol{\eta} \\ &\quad + \tfrac{1}{2}f_3(\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\times\boldsymbol{\eta} + f_7(1 - f_5\phi^2)(\boldsymbol{\phi}\cdot\boldsymbol{\eta})\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\omega}) - \tfrac{1}{2}f_7(\boldsymbol{\phi}\cdot\boldsymbol{\omega})\phi^2\boldsymbol{\phi}\times\boldsymbol{\eta} - \tfrac{1}{2}f_7\phi^2[(\boldsymbol{\phi}\cdot\boldsymbol{\eta})\boldsymbol{\phi}\times\boldsymbol{\omega} - (\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\times\boldsymbol{\eta}] \\ &\quad - f_8(\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\eta})\end{aligned} \tag{A2}$$

Collecting the terms,

$$\begin{aligned}\dot{\boldsymbol{\eta}} &= \mathbf{a}_{SF} + \frac{1}{2}\boldsymbol{\phi}\times\mathbf{a}_{SF} + f_5[\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\mathbf{a}_{SF}) + \boldsymbol{\phi}\times(\boldsymbol{\eta}\times\boldsymbol{\omega})] \\ &\quad + \left[\frac{\sin\phi + \phi}{2\phi^3(1 - \cos\phi)} - \frac{2}{\phi^4}\right](\boldsymbol{\phi}\cdot\boldsymbol{\eta})\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\omega}) + A\boldsymbol{\omega}\times\boldsymbol{\eta} \\ &\quad + B(\boldsymbol{\phi}\times\boldsymbol{\omega})\times\boldsymbol{\eta} + C(\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\times\boldsymbol{\eta} + D(\boldsymbol{\phi}\cdot\boldsymbol{\eta})\boldsymbol{\phi}\times\boldsymbol{\omega} + H\end{aligned} \tag{A3}$$

where

$$A \triangleq -(\tfrac{1}{2} - f_4\phi^2)(1 - f_5\phi^2) - \tfrac{1}{2}f_3\phi^2$$

$$B \triangleq -\tfrac{1}{2}(\tfrac{1}{2} - f_4\phi^2) + f_3(1 - f_5\phi^2)$$

$$C \triangleq -(\tfrac{1}{2} - f_4\phi^2)f_5 + f_3 - f_6$$

$$D \triangleq -\tfrac{1}{2}(f_5 - f_8\phi^2) + (f_6 - \tfrac{1}{2}f_3)(1 - f_5\phi^2) - \tfrac{1}{2}f_7\phi^2 \tag{A4}$$

$$E \triangleq -(f_5 - f_8\phi^2)(1 - f_5\phi^2) + \tfrac{1}{4}f_3\phi^2 + f_5$$

$$F \triangleq -(f_5 - f_8\phi^2)f_5 - \tfrac{1}{4}f_3 - f_8$$

$$G \triangleq \frac{1}{2}f_6 + f_7(1 - f_5\phi^2) - \left[\frac{\sin\phi + \phi}{2\phi^3(1 - \cos\phi)} - \frac{2}{\phi^4}\right]$$

$$\begin{aligned}H &\triangleq E\boldsymbol{\phi}\times(\boldsymbol{\omega}\times\boldsymbol{\eta}) + F(\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\eta}) + G(\boldsymbol{\phi}\cdot\boldsymbol{\eta})\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\omega}) \\ &= (E - G\phi^2)(\boldsymbol{\phi}\cdot\boldsymbol{\eta})\boldsymbol{\omega} - (E + F\phi^2)(\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\eta} + (F + G)(\boldsymbol{\phi}\cdot\boldsymbol{\eta})(\boldsymbol{\phi}\cdot\boldsymbol{\omega})\boldsymbol{\phi}\end{aligned}$$

Substituting Eqs. (8) and (16) in [1] yields

$$\begin{gathered}A = -1/2, \qquad B = -f_5 \\ C = D = H = (E - G\phi^2) = (E + F\phi^2) = (F + G) = 0\end{gathered} \tag{A5}$$

Therefore, the differential equation for VTV in Eq. (15) in [1] is reduced to exactly Eq. (9) despite the difference in mathematical notations.

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